For maximally monotone linear relations, dense type, negative-infimum type, and Fitzpatrick-Phelps type all coincide with monotonicity of the adjoint

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Abstract

It is shown that, for maximally monotone linear relations defined on a general Banach space, the monotonicities of dense type, of negative-infimum type, and of Fitzpatrick-Phelps type are the same and equivalent to monotonicity of the adjoint. This result also provides affirmative answers to two problems: one posed by Phelps and Simons, and the other by Simons.

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1 Introduction

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. The *domain* of A, written as $\operatorname{dom} A$, is $\operatorname{dom} A = \big\{x \in X \mid Ax \neq \varnothing\big\}$ and $\operatorname{ran} A = A(X)$ for the *range* of A. Recall that A is *monotone* if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A\ \forall (y,y^*) \in \operatorname{gra} A, \tag{1}$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say (x,x^*) is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

We now define the three aforementioned types of maximally monotone operators.

Definition 1.1 *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then three key types of monotone operators are defined as follows.*

(i) *A is* of dense type or type (D) *(see [21]) if for every $(x^{**},x^*) \in X^{**} \times X^*$ with*

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^*\rangle \geq 0,$$

there exist a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that (a_α, a^*_α) weak×strong converges to (x^{**},x^*).*

(ii) *A is* of type negative infimum (NI) *(see [30]) if*

$$\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle a, x^*\rangle + \langle a^*, x^{**}\rangle - \langle a, a^*\rangle\big) \geq \langle x^{**}, x^*\rangle, \quad \forall (x^{**},x^*) \in X^{**} \times X^*.$$

(iii) *A is* of type Fitzpatrick-Phelps (FP) *(see [20]) if for every open convex subset U of X^* such that $U \cap \operatorname{ran} A \neq \varnothing$, the implication*

$$x^* \in U \text{ and } (x,x^*) \in X\times X^* \text{ is monotonically related to } \operatorname{gra} A \cap (X \times U) \Rightarrow (x,x^*) \in \operatorname{gra} A$$

holds.

We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. By saying $A : X \rightrightarrows X^*$ is *at most single-valued*, we mean that for every $x \in X$, Ax is either a singleton or empty. In this case, we follow a slight but common abuse of notation and write $A\colon \operatorname{dom} A \to X^*$. Conversely, if $T\colon D \to X^*$, we may identify T with $A : X \rightrightarrows X^*$, where A is at most single-valued with $\operatorname{dom} A = D$.

Monotone operators have proven to be a key class of objects in both modern Optimization and Analysis; see, e.g., [10, 11, 12], the books [3, 14, 18, 26, 31, 33, 29, 42] and the references therein.

In this paper, we provide tools to give affirmative answers to two questions respectively posed by Phelps and Simons, and by Simons. Phelps and Simons posed the following question in [27, Section 9, item 2]: *Let $A : \operatorname{dom} A \to X^*$ be linear and maximally monotone. Assume that A^* is monotone. Is A necessarily of type (D)*?

Simons posed another question in [33, Problem 47.6]: *Let $A : \operatorname{dom} A \to X^*$ be linear and maximally monotone. Assume that A is of type (FP). Is A necessarily of type (NI)?*

We give affirmative answers to the above questions in Theorem 3.1. Moreover, we generalize the results to the linear relations. Linear relations have recently become a center of attention in Monotone Operator Theory; see, e.g., [1, 2, 4, 5, 6, 7, 8, 9, 16, 17, 27, 34, 35, 36, 37, 38, 39, 40, 41] and Cross' book [19] for general background on linear relations.

We adopt standard notation used in these books: Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, and $\overline{C}$ is the *norm closure* of C . The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

For every $x \in X$, the *normal cone* operator of C at x is defined by $N_C(x) = \big\{x^* \in X^* \mid \sup_{c\in C}\langle c - x, x^*\rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. For $x, y \in X$, we set $[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. If Z is a real Banach space with continuous dual Z^* and a subset S of Z, we denote $S^\perp$ by $S^\perp = \big\{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\big\}$. Given a subset D of Z^*, we set $D_\perp = D^\perp \cap Z$. The *adjoint* of A, written as A^*, is defined by

$$\operatorname{gra} A^* = \big\{(x^{**}, x^*) \in X^{**} \times X^* \mid (x^*, -x^{**}) \in (\operatorname{gra} A)^\perp\big\}.$$

Let $f\colon X \to \left]-\infty, +\infty\right]$. Then $\operatorname{dom} f = f^{-1}(\mathbb{R})$ is the *domain* of f, and $f^*\colon X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x\in X}(\langle x, x^*\rangle - f(x))$ is the *Fenchel conjugate* of f. For $\varepsilon \geq 0$, the *ε–subdifferential* of f is defined by $\partial_\varepsilon f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y) + \varepsilon\big\}$. We also set $\partial f = \partial_0 f$.

Let $F : X \times X^* \to \left]-\infty, +\infty\right]$. We say F is a *representative* of a maximally monotone operator $A : X \rightrightarrows X^*$ if F is lower semicontinuous and convex with $F \geq \langle\cdot,\cdot\rangle$ on $X \times X^*$

and

$$\operatorname{gra} A = \{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\}.$$

Let $(z, z^*) \in X \times X^*$. Then $F_{(z,z^*)} : X \times X^* \to \left]-\infty, +\infty\right]$ [25, 33, 23] is defined by

$$\begin{aligned} F_{(z,z^*)}(x, x^*) &= F(z + x, z^* + x^*) - \big(\langle x, z^* \rangle + \langle z, x^* \rangle + \langle z, z^* \rangle\big) \\ &= F(z + x, z^* + x^*) - \langle z + x, z^* + x^* \rangle + \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \end{aligned} \tag{3}$$

Moreover, the *closed unit ball* in X is denoted by $B_X = \big\{x \in X \mid \|x\| \leq 1\big\}$, and $\mathbb{N} = \{1, 2, 3, \ldots\}$. We identify X with its canonical image in the bidual space X^{**}. Furthermore, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are likewise paired via

$$\langle (x, x^*), (y^*, y^{**}) \rangle = \langle x, y^* \rangle + \langle x^*, y^{**} \rangle,$$

where $(x, x^*) \in X \times X^*$ and $(y^*, y^{**}) \in X^* \times X^{**}$. The norm on $X \times X^*$, written as $\|\cdot\|_1$, is defined by $\|(x, x^*)\|_1 = \|x\| + \|x^*\|$ for every $(x, x^*) \in X \times X^*$.

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. The main result (Theorem 3.1) is provided in Section 3. The affirmative answers to Phelps-Simons' and Simons' questions are then apparent.

2 Auxiliary results

Fact 2.1 (Rockafellar) (See [28, Theorem 3(a)], [33, Corollary 10.3] or [42, Theorem 2.8.7(iii)].) *Let $f, g : X \to \left]-\infty, +\infty\right]$ be proper convex functions. Assume that there exists a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. For every $x^* \in X^*$, we have*

$$(f + g)^*(x^*) = \min_{y^* \in X^*} \left[f^*(y^*) + g^*(x^* - y^*)\right].$$

Fact 2.2 (Borwein) (See [13, Theorem 1] or [42, Theorem 3.1.1].) *Let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous and convex function. Let $\varepsilon > 0$ and $\beta \geq 0$ (where $\frac{1}{0} = \infty$). Assume that $x_0 \in \operatorname{dom} f$ and $x_0^* \in \partial_\varepsilon f(x_0)$. There exist $x_\varepsilon \in X, x_\varepsilon^* \in X^*$ such that*

$$\begin{aligned} &\|x_\varepsilon - x_0\| + \beta\,|\langle x_\varepsilon - x_0, x_0^* \rangle| \leq \sqrt{\varepsilon}, \quad x_\varepsilon^* \in \partial f(x_\varepsilon), \\ &\|x_\varepsilon^* - x_0^*\| \leq \sqrt{\varepsilon}(1 + \beta\|x_0^*\|), \quad |\langle x_\varepsilon - x_0, x_\varepsilon^* \rangle| \leq \varepsilon + \frac{\sqrt{\varepsilon}}{\beta}. \end{aligned}$$

Fact 2.3 (Simons) (See [32, Theorem 17] or [33, Theorem 37.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator such that A is of type (D). Then A is type of (FP).*

Fact 2.4 (Simons) (See [33, Lemma 19.7 and Section 22].) *Let $A : X \rightrightarrows X^*$ be a monotone operator such that* $\operatorname{gra} A$ *is convex with* $\operatorname{gra} A \neq \varnothing$*. Then the function*

$$g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^* \rangle + \iota_{\operatorname{gra} A}(x, x^*) \tag{4}$$

is proper and convex.

Fact 2.5 (Marques Alves and Svaiter) (See [24, Theorem 4.4].) *Let $A : X \rightrightarrows X^*$ be maximally monotone, and let $F : X \to \left]-\infty, +\infty\right]$ be a representative of A. Then the following are equivalent.*

(i) *A is type of (D).*

(ii) *A is of type (NI).*

(iii) *For every $(x_0, x_0^*) \in X \times X^*$,*

$$\inf_{(x,x^*) \in X \times X^*} \left[F_{(x_0,x_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] = 0.$$

Remark 2.6 The implication (i)⇒(ii) in Fact 2.5 was first proved by Simons (see [30, Lemma 15] or [33, Theorem 36.3(a)]).

Fact 2.7 (Cross) *Let $A\colon X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(i) $Ax = x^* + A0, \quad \forall x^* \in Ax.$

(ii) $(\forall x^{**} \in \operatorname{dom} A^*)(\forall y \in \operatorname{dom} A)$ $\langle A^* x^{**}, y \rangle = \langle x^{**}, Ay \rangle$ *is a singleton.*

(iii) $(\operatorname{dom} A)^\perp = A^*0$. *If* $\operatorname{gra} A$ *is closed, then* $(\operatorname{dom} A^*)_\perp = A0$.

Proof. (i): See [19, Proposition I.2.8(a)]. (ii): See [19, Proposition III.1.2]. (iii) : See [19, Proposition III.1.4(b)&(d)]. ■

Lemma 2.8 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then* $(\operatorname{dom} A)^\perp = A0 = A^*0 = (\operatorname{dom} A^*)_\perp$.

Proof. (See also [5, Theorem 3.2(iii)] when X is reflexive.) Since $A + N_{\operatorname{dom} A} = A + (\operatorname{dom} A)^\perp$ is a monotone extension of A and A is maximally monotone, we must have $A + (\operatorname{dom} A)^\perp = A$. Then $A0 + (\operatorname{dom} A)^\perp = A0$. As $0 \in A0$, $(\operatorname{dom} A)^\perp \subseteq A0$.

On the other hand, take $x \in \operatorname{dom} A$. Then there exists $x^* \in X^*$ such that $(x, x^*) \in \operatorname{gra} A$. By monotonicity of A and since $(0, A0) \subseteq \operatorname{gra} A$, we have $\langle x, x^* \rangle \geq \sup \langle x, A0 \rangle$. Since $A0$ is a linear subspace, we obtain $x \perp A0$. This implies $A0 \subseteq (\operatorname{dom} A)^\perp$.

Combining the above, we have $(\operatorname{dom} A)^{\perp} = A0$. Thus, by Fact 2.7(iii), $(\operatorname{dom} A)^{\perp} = A0 = A^*0 = (\operatorname{dom} A^*)_{\perp}$. ■

Lemma 2.9 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then $\langle x^{**}, A^*x^{**}\rangle$ is single-valued for every $x^{**} \in \operatorname{dom} A^*$.*

Proof. Take $x^{**} \in \operatorname{dom} A^*$ and $x^* \in A^*x^{**}$. By Fact 2.7(i) and Lemma 2.8,

$$\langle x^{**}, A^*x^{**}\rangle = \langle x^{**}, x^* + A^*0\rangle = \langle x^{**}, x^*\rangle.$$

Thus $\langle x^{**}, A^*x^{**}\rangle$ is single-valued. ■

3 Main result

Theorem 3.1 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then the following are equivalent.*

(i) *A is of type (D).*

(ii) *A is of type (NI).*

(iii) *A^* is monotone.*

(iv) *A is of type (FP).*

Proof. "(i)⇔(ii)": Fact 2.5.

"(ii)⇒(iii)": Suppose to the contrary that there exists $(a_0^{**}, a_0^*) \in \operatorname{gra} A^*$ such that $\langle a_0^{**}, a_0^*\rangle < 0$. Then we have

$$\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle a, -a_0^*\rangle + \langle a_0^{**}, a^*\rangle - \langle a, a^*\rangle\big) = \sup_{(a,a^*)\in \operatorname{gra} A} \{-\langle a, a^*\rangle\} = 0 < \langle -a_0^{**}, a_0^*\rangle,$$

which contradicts that A is type of (NI). Hence A^* is monotone.

"(iii)⇒(ii)": Define

$$F : X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^*\rangle.$$

Since A is maximally monotone, Fact 2.4 implies that F is proper lower semicontinuous and convex, and a representative of A. Let $(v_0, v_0^*) \in X \times X^*$. Recalling (3), note that

$$F_{(v_0,v_0^*)} : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(v_0 + x, v_0^* + x^*) + \langle x, x^*\rangle \tag{5}$$

is proper lower semicontinuous and convex. By Fact 2.1, there exists $(y^{**}, y^*) \in X^{**} \times X^*$ such that

$$\begin{aligned} K &:= \inf_{(x,x^*)\in X\times X^*} \left[F_{(v_0,v_0^*)}(x,x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] \\ &= -\left(F_{(v_0,v_0^*)} + \tfrac{1}{2}\|\cdot\|^2 + \tfrac{1}{2}\|\cdot\|^2\right)^*(0,0) \\ &= -F^*_{(v_0,v_0^*)}(y^*, y^{**}) - \tfrac{1}{2}\|y^{**}\|^2 - \tfrac{1}{2}\|y^*\|^2. \end{aligned} \tag{6}$$

Since $(x, x^*) \mapsto F_{(v_0,v_0^*)}(x,x^*) + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$ is coercive, there exist $M > 0$ and a sequence $(a_n, a_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ such that

$$\|a_n\| + \|a_n^*\| \leq M \tag{7}$$

and

$$\begin{aligned} &F_{(v_0,v_0^*)}(a_n, a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 \\ &< K + \tfrac{1}{n^2} = -F^*_{(v_0,v_0^*)}(y^*, y^{**}) - \tfrac{1}{2}\|y^{**}\|^2 - \tfrac{1}{2}\|y^*\|^2 + \tfrac{1}{n^2} \quad \text{(by (6))} \end{aligned}$$

$$\Rightarrow F_{(v_0,v_0^*)}(a_n, a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*, y^{**}) + \tfrac{1}{2}\|y^{**}\|^2 + \tfrac{1}{2}\|y^*\|^2 < \tfrac{1}{n^2} \tag{8}$$

$$\Rightarrow F_{(v_0,v_0^*)}(a_n, a_n^*) + F^*_{(v_0,v_0^*)}(y^*, y^{**}) + \langle a_n, -y^*\rangle + \langle a_n^*, -y^{**}\rangle < \tfrac{1}{n^2} \tag{9}$$

$$\Rightarrow (y^*, y^{**}) \in \partial_{\frac{1}{n^2}} F_{(v_0,v_0^*)}(a_n, a_n^*) \quad \text{(by [42, Theorem 2.4.2(ii)])}. \tag{10}$$

Set $\beta = \frac{1}{\max\{\|y^*\|,\|y^{**}\|\}+1}$. Then by Fact 2.2, there exist sequences $(\widetilde{a_n}, \widetilde{a_n^*})_{n\in\mathbb{N}}$ in $X \times X^*$ and $(y_n^*, y_n^{**})_{n\in\mathbb{N}}$ in $X^* \times X^{**}$ such that

$$\|a_n - \widetilde{a_n}\| + \|a_n^* - \widetilde{a_n^*}\| + \beta\left|\langle \widetilde{a_n} - a_n, y^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y^{**}\rangle\right| \leq \tfrac{1}{n} \tag{11}$$

$$\max\{\|y_n^* - y^*\|, \|y_n^{**} - y^{**}\|\} \leq \tfrac{2}{n} \tag{12}$$

$$\left|\langle \widetilde{a_n} - a_n, y_n^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y_n^{**}\rangle\right| \leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} \tag{13}$$

$$(y_n^*, y_n^{**}) \in \partial F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}), \quad \forall n \in \mathbb{N}. \tag{14}$$

Then we have

$$\begin{aligned} &\langle \widetilde{a_n}, y_n^*\rangle + \langle \widetilde{a_n^*}, y_n^{**}\rangle - \langle a_n, y^*\rangle - \langle a_n^*, y^{**}\rangle \\ &= \langle \widetilde{a_n} - a_n, y_n^*\rangle + \langle a_n, y_n^* - y^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y_n^{**}\rangle + \langle a_n^*, y_n^{**} - y^{**}\rangle \\ &\leq \left|\langle \widetilde{a_n} - a_n, y_n^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y_n^{**}\rangle\right| + |\langle a_n, y_n^* - y^*\rangle| + |\langle a_n^*, y_n^{**} - y^{**}\rangle| \\ &\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \|a_n\| \cdot \|y_n^* - y^*\| + \|a_n^*\| \cdot \|y_n^{**} - y^{**}\| \quad \text{(by (13))} \\ &\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + (\|a_n\| + \|a_n^*\|) \cdot \max\{\|y_n^* - y^*\|, \|y_n^{**} - y^{**}\|\} \\ &\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \tfrac{2}{n}M \quad \text{(by (7) and (12))}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{15}$$

By (11), we have

$$\big|\|a_n\| - \|\widetilde{a_n}\|\big| + \big|\|a_n^*\| - \|\widetilde{a_n^*}\|\big| \leq \tfrac{1}{n}. \tag{16}$$

Thus by (7), we have

$$\begin{aligned}
&\big|\|a_n\|^2 - \|\widetilde{a_n}\|^2\big| + \big|\|a_n^*\|^2 - \|\widetilde{a_n^*}\|^2\big| \\
&= \big|\|a_n\| - \|\widetilde{a_n}\|\big|\big(\|a_n\| + \|\widetilde{a_n}\|\big) + \big|\|a_n^*\| - \|\widetilde{a_n^*}\|\big|\left(\|a_n^*\| + \|\widetilde{a_n^*}\|\right) \\
&\leq \tfrac{1}{n}\left(2\|a_n\| + \tfrac{1}{n}\right) + \tfrac{1}{n}\left(2\|a_n^*\| + \tfrac{1}{n}\right) \quad \text{(by (16))} \\
&\leq \tfrac{1}{n}(2M + \tfrac{2}{n}) = \tfrac{2}{n}M + \tfrac{2}{n^2}, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{17}$$

Similarly, by (12), for all $n \in \mathbb{N}$, we have

$$\big|\|y_n^*\|^2 - \|y^*\|^2\big| \leq \tfrac{4}{n}\|y^*\| + \tfrac{4}{n^2} \leq \tfrac{4}{n\beta} + \tfrac{4}{n^2}, \quad \big|\|y_n^{**}\|^2 - \|y^{**}\|^2\big| \leq \tfrac{4}{n}\|y^{**}\| + \tfrac{4}{n^2} \leq \tfrac{4}{n\beta} + \tfrac{4}{n^2}. \tag{18}$$

Thus

$$\begin{aligned}
&F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 + \tfrac{1}{2}\|y_n^{**}\|^2 \\
&= \Big[F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 + \tfrac{1}{2}\|y_n^{**}\|^2\Big] \\
&\quad - \Big[F_{(v_0,v_0^*)}(a_n, a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*, y^{**}) + \tfrac{1}{2}\|y^{**}\|^2 + \tfrac{1}{2}\|y^*\|^2\Big] \\
&\quad + \Big[F_{(v_0,v_0^*)}(a_n, a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*, y^{**}) + \tfrac{1}{2}\|y^{**}\|^2 + \tfrac{1}{2}\|y^*\|^2\Big] \\
&< \Big[F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}) - F_{(v_0,v_0^*)}(a_n, a_n^*) - F^*_{(v_0,v_0^*)}(y^*, y^{**})\Big] \\
&\quad + \tfrac{1}{2}\left[\|\widetilde{a_n}\|^2 + \|\widetilde{a_n^*}\|^2 - \|a_n\|^2 - \|a_n^*\|^2\right] \\
&\quad + \tfrac{1}{2}\left[\|y_n^*\|^2 + \|y_n^{**}\|^2 - \|y^{**}\|^2 - \|y^*\|^2\right] + \tfrac{1}{n^2} \quad \text{(by (8))} \\
&\leq \left[\langle \widetilde{a_n}, y_n^*\rangle + \langle \widetilde{a_n^*}, y_n^{**}\rangle - \langle a_n, y^*\rangle - \langle a_n^*, y^{**}\rangle\right] \quad \text{(by (14))} \\
&\quad + \tfrac{1}{2}\left(\big|\|\widetilde{a_n}\|^2 - \|a_n\|^2\big| + \big|\|\widetilde{a_n^*}\|^2 - \|a_n^*\|^2\big|\right) \\
&\quad + \tfrac{1}{2}\left(\big|\|y_n^*\|^2 - \|y^*\|^2\big| + \big|\|y_n^{**}\|^2 - \|y^{**}\|^2\big|\right) + \tfrac{1}{n^2} \\
&\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \tfrac{2}{n}M + \tfrac{1}{n}M + \tfrac{1}{n^2} + \tfrac{4}{n\beta} + \tfrac{4}{n^2} + \tfrac{1}{n^2} \quad \text{(by (15), (17) and (18))} \\
&= \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{19}$$

By (14), (5), and [42, Theorem 3.2.4(vi)&(ii)], there exists a sequence $(z_n^*, z_n^{**})_{n\in\mathbb{N}}$ in $(\operatorname{gra} A)^\perp$ and such that

$$(y_n^*, y_n^{**}) = (\widetilde{a_n^*}, \widetilde{a_n}) + (z_n^*, z_n^{**}), \quad \forall n \in \mathbb{N}. \tag{20}$$

Since A^* is monotone and $(z_n^{**}, z_n^*) \in \operatorname{gra}(-A^*)$, it follows from (20) that

$$\begin{aligned}
&\langle y_n^*, y_n^{**}\rangle - \langle y_n^*, \widetilde{a_n}\rangle - \langle y_n^{**}, \widetilde{a_n^*}\rangle + \langle \widetilde{a_n^*}, \widetilde{a_n}\rangle = \langle y_n^* - \widetilde{a_n^*}, y_n^{**} - \widetilde{a_n}\rangle = \langle z_n^*, z_n^{**}\rangle \leq 0 \\
&\Rightarrow \langle y_n^*, y_n^{**}\rangle \leq \langle y_n^*, \widetilde{a_n}\rangle + \langle y_n^{**}, \widetilde{a_n^*}\rangle - \langle \widetilde{a_n^*}, \widetilde{a_n}\rangle, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Then by (5) and (14), we have $\langle \widetilde{a_n^*}, \widetilde{a_n}\rangle = F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*})$ and

$$\langle y_n^*, y_n^{**}\rangle \leq \langle y_n^*, \widetilde{a_n}\rangle + \langle y_n^{**}, \widetilde{a_n^*}\rangle - F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) = F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}), \quad \forall n \in \mathbb{N}. \tag{21}$$

By (19) and (21), we have

$$\begin{aligned} & F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \langle y_n^*, y_n^{**}\rangle + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 + \tfrac{1}{2}\|y_n^{**}\|^2 < \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M \\ & \Rightarrow F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 < \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{22}$$

Thus by (22),

$$\inf_{(x,x^*)\in X\times X^*} \left[F_{(v_0,v_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] \leq 0. \tag{23}$$

By (5),

$$\inf_{(x,x^*)\in X\times X^*} \left[F_{(v_0,v_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] \geq 0. \tag{24}$$

Combining (23) with (24), we obtain

$$\inf_{(x,x^*)\in X\times X^*} \left[F_{(v_0,v_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] = 0. \tag{25}$$

Thus by Fact 2.5, A is of type (NI). This concludes the proof that (i), (ii), and (iii) coincide.

Now "(i)⇒(iv)" follows from Fact 2.3. It remains to show only:

"(iv)⇒(iii)": Let $(x_0^{**}, x_0^*) \in \operatorname{gra} A^*$. We must show that

$$\langle x_0^{**}, x_0^*\rangle \geq 0. \tag{26}$$

We can and do assume that

$$\langle x_0^{**}, x_0^*\rangle \neq 0. \tag{27}$$

By Fact 2.7(ii),

$$\langle x_0^{**}, Aa\rangle = \langle x_0^*, a\rangle, \quad \forall a \in \operatorname{dom} A. \tag{28}$$

We claim that there exists $a_0 \in \operatorname{dom} A$ such that

$$\langle x_0^*, a_0\rangle < 0. \tag{29}$$

Recalling that $\operatorname{dom} A$ is a subspace, we suppose to the contrary that

$$\langle x_0^*, a\rangle = 0, \quad \forall a \in \operatorname{dom} A. \tag{30}$$

Thus

$$(0, x_0^*) \in \operatorname{gra} A^*. \tag{31}$$

Since $(x_0^{**}, x_0^*) \in \operatorname{gra} A^*$, $(x_0^{**}, 0) \in \operatorname{gra} A^*$. Thus, by Lemma 2.9,

$$\langle x_0^{**}, x_0^* \rangle = \langle x_0^{**}, 0 \rangle = 0, \tag{32}$$

which contradicts (27). Hence (29) holds. Take $a_0^* \in X^*$ such that $(a_0, a_0^*) \in \operatorname{gra} A$. Set

$$C_n = [a_0^*, x_0^*] + \tfrac{1}{n} B_{X^*}. \tag{33}$$

Then C_n is weak* compact, convex, and $x_0^* \in \operatorname{int} C_n$.

Now we show that

$$(0, x_0^*) \notin \operatorname{gra} A. \tag{34}$$

Suppose to the contrary that $(0, x_0^*) \in \operatorname{gra} A$. By Lemma 2.8, $(0, x_0^*) \in \operatorname{gra} A^*$. Since $(x_0^{**}, x_0^*) \in \operatorname{gra} A^*$, $(x_0^{**}, 0) \in \operatorname{gra} A^*$. Thus by Lemma 2.9 again, we have

$$\langle x_0^{**}, x_0^* \rangle = \langle x_0^{**}, 0 \rangle = 0, \tag{35}$$

which contradicts (27). Thus (34) holds.

By (33), $x_0^* \in \operatorname{int} C_n$. Then by (34), $a_0^* \in \operatorname{ran} A \cap \operatorname{int} C_n$ and that A is of type (FP), we have

$$\begin{aligned} 0 &> \inf_{(a,a^*) \in X \times X^*} \big(-\langle x_0^*, a \rangle + \langle a, a^* \rangle + \iota_{\operatorname{gra} A}(a, a^*) + \iota_{X \times C_n}(a, a^*) \big) \\ &= -\left[\langle \cdot, \cdot \rangle + \iota_{\operatorname{gra} A} + \iota_{X \times C_n} \right]^* (x_0^*, 0), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{36}$$

By Fact 2.4,

$$F \colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^* \rangle + \iota_{\operatorname{gra} A}(x, x^*) \quad \text{is proper and convex .} \tag{37}$$

Since

$$(a_0, a_0^*) \in \operatorname{gra} A \quad \text{and} \quad a_0^* \in \operatorname{ran} A \cap \operatorname{int} C_n, \quad \forall n \in \mathbb{N}, \tag{38}$$

$(a_0, a_0^*) \in \operatorname{dom} F \cap \operatorname{int} \operatorname{dom} \iota_{X \times C_n}$. Then

$$\iota_{X \times C_n} \text{ is continuous at } (a_0, a_0^*), \quad \forall n \in \mathbb{N}. \tag{39}$$

Using (36), (39), (37), Fact 2.1, and the fact that $(x_0^{**}, x_0^*) \in \operatorname{gra} A^* \Leftrightarrow F^*(x_0^*, -x_0^{**}) = 0$, we have

$$\begin{aligned}
0 &> - \min_{(y^{**},y^*)\in X^{**}\times X^*} \left[F^*(x_0^* + y^*, y^{**}) + \iota^*_{X\times C_n}(-y^*, -y^{**})\right] \\
&\geq - \left[F^*(x_0^*, -x_0^{**}) + \iota^*_{X\times C_n}(0, x_0^{**})\right] \\
&= -\iota^*_{X\times C_n}(0, x_0^{**}) \\
&= -\tfrac{1}{n}\|x_0^{**}\| - \max\{\langle x_0^*, x_0^{**}\rangle, \langle x_0^{**}, a_0^*\rangle\}. \qquad (40)
\end{aligned}$$

Take $n \to \infty$ in (40) to get

$$\max\{\langle x_0^*, x_0^{**}\rangle, \langle x_0^{**}, a_0^*\rangle\} \geq 0. \qquad (41)$$

Since

$$\langle x_0^{**}, a_0^*\rangle = \langle x_0^*, a_0\rangle < 0, \quad \text{(by (28) and (29))} \qquad (42)$$

it follows from (41) that

$$\langle x_0^{**}, x_0^*\rangle \geq 0. \qquad (43)$$

Thus (26) holds and hence A^* is monotone. This establishes (iii) as required. ■

Remark 3.2 When A is linear and continuous, Theorem 3.1 is due to Bauschke and Borwein [1, Theorem 4.1]. Phelps and Simons in [27, Theorem 6.7] considered the case when A is linear but possibly discontinuous; they arrived at some of the implications of Theorem 3.1 in that case.

(i) The proof of (ii)⇒(iii) in Theorem 3.1 follows closely that of [15, Theorem 2].

(ii) Theorem 3.1(iii)⇒(i) gives an affirmative answer to a problem posed by Phelps and Simons in [27, Section 9, item 2] on the converse of [27, Theorem 6.7(c)⇒(f)].

(iii) Theorem 3.1(iv)⇒(ii) gives an affirmative answer to a problem posed by Simons in [33, Problem 47.6].

(iv) The proof of (iii)⇒(ii) in Theorem 3.1 was partially inspired by that of [43, Theorem 32.L] and that of [22, Theorem 2.1].

(v) The proof of (iv)⇒(iii) in Theorem 3.1 closely follows that of [1, Theorem 4.1(iv)⇒(v)].

We conclude with an application of Theorem 3.1 to an operator studied previously by Phelps and Simons [27].

Example 3.3 Suppose that $X = L^1[0,1]$ so that $X^* = L^\infty[0,1]$, let

$$D = \big\{x \in X \mid x \text{ is absolutely continuous}, x(0) = 0, x' \in X^*\big\},$$

and set

$$A\colon X \rightrightarrows X^*\colon x \mapsto \begin{cases} \{x'\}, & \text{if } x \in D; \\ \varnothing, & \text{otherwise.} \end{cases}$$

By [27, Example 4.3], A is an at most single-valued maximal monotone linear relation with proper dense domain, and A is neither symmetric nor skew. Moreover,

$$\operatorname{dom} A^* = \{z \in X^{**} \mid z \text{ is absolutely continuous}, z(1) = 0, z' \in X^*\} \subseteq X$$

$A^*z = -z', \forall z \in \operatorname{dom} A^*$, and A^* is monotone. Therefore, Theorem 3.1 implies that A is of type of (D), of type (NI), and of type (FP).

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